Instability of multiple pulses in coupled nonlinear Schrödinger equations

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An analytical technique for constructing multiple pulses is presented in this article; with it we uncover a homoclinic bifurcation through which multihumped solitary waves can be generated in systems of coupled nonlinear Schrödinger equations. A method is then developed to determine the (in)stability of multiple pulses produced by this mechanism. The analysis is applied to two models that describe optical phenomena in dispersive quadratic and Kerr media, respectively. It sheds considerable light upon the characteristics that predispose multiple pulses arising in this class of systems to be unstable.

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I. INTRODUCTION

In the study of pulse propagation in birefringent optical communication lines, or wave interaction in materials with nonlinear response of various kinds, models that take the form of two or more Schrödinger equations coupled together in a nonlinear fashion are encountered frequently. Examples include the systems that describe birefringence effects in Kerr ($\chi^{(3)}$) materials [1,2], incoherent beam interaction in photorefractive (saturable) media [3], parametric wave interaction and second-harmonic generation [4–6], phase-matched third-harmonic generation in Kerr media [7], three-wave interaction in $\chi^{(2)}$ (quadratic) media [8,9], or $\chi^{(2)}+\chi^{(3)}$ competing nonlinearities [10].

After suitable normalization, such models are written as systems of partial differential equations (PDEs),

$$i\sigma_j\frac{\partial u_j}{\partial z}+s_j\Delta u_j-\theta_j u_j+f_j(u)=0, \tag{1}$$

where $j=1,\dots,m$, $\sigma_j>0$, $s_j=\pm 1$, and $u=(u_1,\dots,u_m)$ with the $u_j(z,\boldsymbol{t})\in\mathbb{C}$. The independent variable $z\in\mathbb{R}$ measures propagation distance, i.e., distance along the optical fiber or longitudinal distance in a waveguide; Δ denotes the Laplacian operator in $\boldsymbol{t}$, where $\boldsymbol{t}\in\mathbb{R}^n$ represents retarded time (with $n=1$) or the transverse spatial variable (with $n=1,2$). Furthermore, these systems are generally conservative and expressible in the abstract infinite-dimensional Hamiltonian form

$$\frac{\partial u}{\partial z}=\mathcal{J}\mathcal{H}'(u), \tag{2}$$

where $\mathcal{J}$ is a skew-symmetric matrix, $\mathcal{H}$ is a functional of u, and the prime denotes the variational derivative.

A variety of solitary waves—standing or traveling waves that are localized in the $\boldsymbol{t}$ direction(s)—have been found in many coupled nonlinear Schrödinger (NLS) equations. In particular, multihumped vector soliton bound states were observed in numerical simulations of various systems, and are widely believed to be unstable in general. In this article, we present a powerful analytical method for finding multipulses and for studying their stability. A mechanism for the generation of widely spaced multihumped waves is identified, which arises from a ‘‘resonant semisimple’’ eigenvalue configuration of the linearized steady-state equations. Information gleaned from the existence analyses then feeds into the stability calculations, which indicate clearly that multipulses are not likely to be stable in conservative coupled NLS systems. We apply the procedure to two specific examples, namely, multisoliton bound states in $\chi^{(2)}$ second-harmonic generation and in isotropic Kerr media; conditions for the generation of multipulses will be uncovered, and the instability of these waves is proved completely.

II. PRELIMINARY PROPERTIES

The special structure possessed by coupled NLS systems of the type (1) will be crucial to the analysis that follows. In this section we state some important properties of, and assumptions on, the system (1). For simplicity, let us restrict our attention to $n=1$, i.e., we consider pulse propagation in optical fibers or self-guided electromagnetic beams in slab waveguides. We shall also take $s_j=+1$ for all j (anomalous dispersion regime). Specific examples are considered in Sec. V below.

The simplest solitary waves are real-valued steady states of Eq. (1); they satisfy ordinary differential equations (ODEs) which can be written as a first-order system

$$u'=p,\quad p'=\Theta u-f(u), \tag{3}$$

where u is now taken to be real valued, $\Theta=\mathrm{diag}(\theta_1,\dots,\theta_m)$, and the primes denote differentiation with respect to the scalar variable t. It is not hard to check the following properties.

(i) The fact that Eq. (1) is a Hamiltonian PDE and, moreover, does not contain any first-order derivatives in t guarantees that Eq. (3) is also Hamiltonian—with $H(u,p)$ being the conserved quantity for the ODEs, say. This fact also implies the next point.

(ii) The system (3) is reversible in t, i.e., it is invariant under the transformation $t \mapsto -t$ and $(u,p) \mapsto (u,-p)$; the latter mapping can be represented by a $2m \times 2m$ diagonal matrix R whose first m entries are $+1$ and whose remaining m entries are -1.

(iii) The linearization of Eq. (3) about the origin produces a matrix $J\nabla^2 H(0)$, where $\nabla^2 H(0) = \mathrm{diag}(-\theta_1, \ldots, -\theta_m, 1, \ldots, 1)$ and

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

is the usual symplectic matrix; we emphasize that the skew-symmetric operators J and $\mathcal{J}$ for the ODE (3) and the PDE (2), respectively, are in general not the same matrices. The eigenvalues of $J\nabla^2 H(0)$ are $\pm\sqrt{\theta_j}$. If all the $\theta_j > 0$, which we shall assume, then zero is a hyperbolic equilibrium of saddle type. Moreover, the eigenvectors $\epsilon_j^{\pm}$ corresponding to $\pm\sqrt{\theta_j}$ are distinct and linearly independent for all values of $\theta_j > 0$, even when eigenvalues overlap; 1 appears as the jth entry and $\pm\sqrt{\theta_j}$ as the $(m+j)$th entry of $\epsilon_j^{\pm}$, and all other entries are zero.

The following assumptions will also be needed.

(i) The system (3) is invariant under a discrete group G of *symmetries*, and each symmetry commutes with both R and $\nabla^2 H(0)$. More specifically, coupled NLS systems often have $\mathbb{Z}_2$ symmetries that can be expressed as matrices of the form

$$\begin{pmatrix} d_i & 0 \\ 0 & d_i \end{pmatrix}, \tag{4}$$

where the d_i are diagonal $m \times m$ matrices whose nonzero entries are ± 1.

(ii) At some special parameter value, say $\mu = 0$, the following properties are assumed to apply. (a) System (3) has a solution $\overline{q}(t)$ which is homoclinic to the origin, i.e., a ''primary'' pulse (usually single-humped) asymptotic to the zero state. By virtue of symmetry, $\{S\overline{q}(t) : S \in G\}$ then forms a set of homoclinic orbits at $\mu = 0$. (b) The orbit $\overline{q}(t)$ is nondegenerate, i.e., the only bounded solution to the equation of variations of Eq. (3) about $\overline{q}(t)$ is $\overline{q}'(t)$. This fact, in conjunction with the conservative property of Eq. (3), implies persistence of the primary pulse(s) for μ close to zero. The primary pulses persisting for $\mu \neq 0$ will be denoted $q(t,\mu)$. (c) The leading eigenvalues of $J\nabla^2 H(0)$ are in resonant semisimple configuration. This means that the eigenvalues whose real parts are of smallest size consist of two points with equal modulus lying on the real line, and each point is actually composed of k overlapping eigenvalues (with $k \geq 2$). The algebraic and geometric multiplicities are the same at these points, due to the properties of the eigenvectors $\epsilon_j^{\pm}$ described above, whence the description ''semisimple.'' This special spectral configuration allows us to derive the expansions

$$q(t,\mu) = \sum_{j=1}^{k} [b_j(\mu)\epsilon_j^{\pm} e^{-\sqrt{\theta_j}|t|}] + \text{higher-order terms} \tag{5}$$

for $|t| \gg 1$. The constants $\overline{b_j}(0)$ can be determined by the asymptotic expansion of $\overline{q}(t)$. We shall write $\nabla^2 H(0)$ as $\nabla^2 H(0,\mu)$ in the future, since it will depend on μ.

Separating Eq. (1), or equivalently Eq. (2), into real and imaginary parts and linearizing about a real z-independent solution $u = \Phi(t)$ gives an operator of the form

$$\mathcal{L} = \mathcal{J}\mathcal{H}''(\Phi) = \mathcal{J}\begin{pmatrix} L_R & 0 \\ 0 & L_I \end{pmatrix}, \tag{6}$$

where L_R and L_I are self-adjoint $m \times m$ matrix differential operators; they depend on Φ.

Let $n(L)$ denote the number of eigenvalues of a linear operator L that have negative real part. Features of the PDEs (1) that will be relevant to our stability analysis are the following.

(i) Invariance under translation in t: $t \mapsto t + t_0$ for any constant t_0. This symmetry generates an integral that is invariant under z evolution, known as ''momentum''; it also contributes a zero eigenvalue to L_R, with corresponding eigenfunction $\Phi'(t) \in \ker(L_R)$.

(ii) Invariance under some phase (rotational) transformation: $(u_1, \ldots, u_m) \mapsto (e^{i\gamma_1 s}u_1, \ldots, e^{i\gamma_m s}u_m)$ for any $s \in \mathbb{R}$, where the γ_j are constants. This symmetry generates an invariant integral known as the ''power'' or ''Manley-Rowe invariant''; it also contributes a zero eigenvalue to L_I, with corresponding eigenfunction $\Gamma\Phi(t) \in \ker(L_R)$, where $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_m)$.

(iii) The essential spectra of L_R and L_I lie on the positive real line and are bounded away from zero. At the primary pulse, $n(L_R) > n(L_I)$. This hypothesis usually holds true for coupled NLS systems, with $n(L_R) = 1$ and $n(L_I) = 0$. One way to show this is by diagonalizing L_R and L_I and using Sturm-Liouville theory together with the previous two properties of Eq. (1) stated above. We remark that, due to Eq. (6), we have $n(\mathcal{H}''(\Phi)) = n(L_R) + n(L_I)$. Thus, $n(L_R) = 1$ and $n(L_I) = 0$ imply $n(\mathcal{H}''(\Phi)) = 1$, which reflects the fact that the primary pulse can be characterized as a ''mountain-pass'' critical point of the functional $\mathcal{H}(u)$ and has Morse index equal to 1: there is one direction near the pulse along which the energy $\mathcal{H}$ decreases.

III. HOMOCLINIC BIFURCATION

We shall construct multipulses by concatenating several copies of the primary pulses, i.e., we seek ''N-pulse'' solutions of Eq. (3) that follow the primary ''1-pulses'' $\{S\overline{q}(t) : S \in G\}$ in a given order $\{\kappa_i\}_{i=1,\ldots,N}$ in phase space, where $\kappa_i \in G$. In particular, the idea is to patch together piecewise continuous solutions that lie close to the 1-pulses by analyzing a set of homoclinic bifurcation equations. The framework, developed by Hale [25], Lin [11], and Sandstede [12] (sometimes referred to as homoclinic Lyapunov-Schmidt reduction, or the HLS method) is briefly described below. Suppose, in a slightly more general context, that at parameter value $\mu = 0$ there exists a nondegenerate heteroclinic cycle connecting some hyperbolic equilibria $P_1, \ldots, P_M$. Let Σ_i be codimension-1 transverse sections to the orbits at $t = 0$. When $\mu \neq 0$, these heteroclinic orbits would typically not persist; however, we can still always find

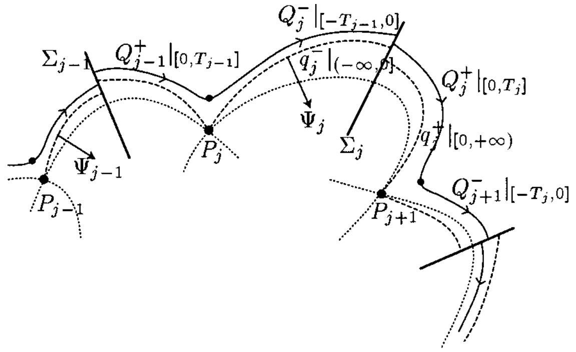

FIG. 1. Piecewise continuous orbits $Q_j^{\pm}$ at $\mu\approx 0$. The solutions $q_j^{\pm}$ are shown with dashed curves, and the trace of the original heteroclinic cycle with dotted curves.

solutions $\{q_i^{\pm}(t,\mu)\}_{i=1,\ldots,M}$ that lie in the stable or unstable manifolds of the P_i; moreover, it is possible to restrict any discontinuities to lie along one particular direction within each transverse section. The jump sizes for the $q_i^{\pm}(t,\mu)$ in Σ_i will be denoted $\xi_i^{\infty}(\mu)$. It can also be shown that for each i there is a unique bounded $\Psi_i(t,\mu)$ that has the property of being perpendicular to the tangent spaces of both the stable and unstable manifolds. Next, it was proved [12] that with $|\mu|$ small, choosing any sequence $\boldsymbol{T}=\{T_j\}_{j\in\mathbb{Z}}$ such that the T_j are all sufficiently large, there exists a unique set of piecewise trajectories $\{Q_j^{\pm}(t,\mu)\}_{j\in\mathbb{Z}}$ lying close to the $q_i^{\pm}(t,\mu)$, and whose times of flight between consecutive sections are specified by the sequence $\{2T_j\}_{j\in\mathbb{Z}}$. Discontinuities only occur at the sections, and time is parametrized so that the sections Σ_i are always reached by the $Q_j^{\pm}$ at exactly $t=0$. The jump sizes are given by the formula

$$\xi_j(\boldsymbol{T},\mu)=\xi_j^{\infty}(\mu)+\langle\Psi_j(-T_{j-1},\mu),q_{j-1}^{+}(T_{j-1},\mu)\rangle -\langle\Psi_j(T_j,\mu),q_{j+1}^{-}(-T_j,\mu)\rangle+\mathcal{R}_j(\boldsymbol{T},\mu), \quad (7)$$

where $\mathcal{R}_j$ is a remainder term. Figure 1 illustrates this setup. To see whether interesting orbits exist when μ is close to zero, one would attempt to solve the system of algebraic equations $\xi_j(\boldsymbol{T},\mu)=0$ for all j with some special type of sequence $\{T_j\}_{j\in\mathbb{Z}}$.

Since here we are interested in homoclinic solutions, we consider only one equilibrium point, i.e., $P_1=\cdots=P_M=0$, and the "heteroclinic cycle" becomes a collection of homoclinic loops $\{S\overline{q}(t)\colon S\in G\}$. To look for N-pulses, set $T_0=T_N=\infty$, while $T_1,\ldots,T_{N-1}$ are assumed to be finite. We then wish to solve

$$\xi_j^{\infty}(\mu)+\langle\Psi_j(-T_{j-1},\mu),q_{j-1}^{+}(T_{j-1},\mu)\rangle -\langle\Psi_j(T_j,\mu),q_{j+1}^{-}(-T_j,\mu)\rangle+\mathcal{R}_j(\boldsymbol{T},\mu)=0 \quad (8)$$

for $j=1,\ldots,N$. While this system of equations seems complicated, it can be simplified considerably by making use of the abundant structure inherent in systems of coupled NLS equations, as described in Sec. II. First, in this case the 1-pulses *persist* for $\mu\neq 0$. This means that $\xi_j^{\infty}(\mu)=0$ and $q_j^{\pm}(t,\mu)=\kappa_j q(t,\mu)$ where $\kappa_j\in G$. Now let us consider the inner product terms; note that they represent information coming from the tails of the 1-pulses, i.e., from near the equilibrium point. Reversibility allows us to write $q(-t,\mu)=Rq(t,\mu)$, and the Hamiltonian structure provides a natural candidate for $\Psi_j(t,\mu)$, namely, $\boldsymbol{\nabla}H(\kappa_j q(t,\mu),\mu)$, which can be expanded as $\nabla^2H(0,\mu)\kappa_j q(t,\mu)+O(|q(t,\mu)|^2)$. Furthermore, we only need to consider j running from 1 to $(N-1)$, because, due to the Hamiltonian structure, $\xi_j(\boldsymbol{T},\mu)=0$ for $j=1,\ldots,(N-1)$ implies $\xi_N(\boldsymbol{T},\mu)=0$ as well [13]. Next, we substitute the asymptotic expansion (5) for $q(t,\mu)$. This expansion is also needed, together with some expressions [12,13] for $\mathcal{R}_j$, to derive estimates for all the remainder terms.

For clarity, from now on we shall concentrate on the case of two coupled nonlinear Schrödinger equations, i.e., $m=2$. Without loss of generality, also assume $\theta_1=1$ (this can be obtained by a rescaling); then a natural choice for our bifurcation parameter is $\mu=\sqrt{\theta_2}-1$, the difference between the eigenvalues of the linearization. Putting together the information so far, we obtain the bifurcation equations

$$\langle M_{j-1}(\mu)\vec{v}(T_{j-1},\mu),\vec{v}(T_{j-1},\mu)\rangle -\langle M_j(\mu)\vec{v}(T_j,\mu),\vec{v}(T_j,\mu)\rangle +O(e^{-\lambda^sT}(e^{-2\lambda^sT_{j-1}}+e^{-2\lambda^sT_j}))=0, \quad (9)$$

where $\lambda_s=\min\{\sqrt{\theta_1},\sqrt{\theta_2}\}=\min\{1,(1+\mu)\}$, $T=\min\{T_k\colon 1\leqslant k\leqslant N-1\}$, $M_j(\mu)=\nabla^2H(0,\mu)R\kappa_j\kappa_{j+1}$, and $\vec{v}(T_j,\mu)$ is the vector whose components are, ordered from first to fourth, $b_1(\mu)e^{-T_j}$, $b_2(\mu)e^{-(1+\mu)T_j}$, $b_1(\mu)e^{-T_j}$, and $-(1+\mu)b_2(\mu)e^{-(1+\mu)T_j}$. If each symmetry κ_j is diagonal with ± 1 as entries, as described in Eq. (4), then $\kappa_j\kappa_{j+1}=\mathrm{diag}(\delta_j^{(1)},\delta_j^{(2)},\delta_j^{(1)},\delta_j^{(2)})$ where $\delta_j^{(l)}=\pm 1$ for $l=1,2$. The bifurcation equations then become

$$\begin{aligned}\xi_j(\boldsymbol{T},\mu)={}&-2[\delta_{j-1}^{(1)}b_1(\mu)^2e^{-2T_{j-1}}\\&+\delta_{j-1}^{(2)}(1+\mu)^2b_2(\mu)^2e^{-2(1+\mu)T_{j-1}}]\\&+2[\delta_j^{(1)}b_1(\mu)^2e^{-2T_j}\\&+\delta_j^{(2)}(1+\mu)^2b_2(\mu)^2e^{-2(1+\mu)T_j}]\\&+O(e^{-\lambda^sT}(e^{-2\lambda^sT_{j-1}}+e^{-2\lambda^sT_j}))=0\end{aligned} \quad (10)$$

for $j=1,\ldots,N-1$. Remember that $T_0=T_N=\infty$, and observe that solving Eq. (10) for $j=0,\ldots,N-1$ is equivalent to solving

$$\begin{aligned}\tfrac{1}{2}(\xi_1+\cdots+\xi_k)={}&\delta_k^{(1)}b_1(\mu)^2e^{-2T_k}\\&+\delta_k^{(2)}(1+\mu)^2b_2(\mu)^2e^{-2(1+\mu)T_k}\\&+O(e^{-3\lambda^sT})=0\end{aligned} \quad (11)$$

for $k=0,\ldots,N-1$. An important point to note is that, if $\delta_k^{(1)}=\delta_k^{(2)}$, then the leading order terms on the left hand side of Eq. (11) will be either strictly positive or strictly negative; therefore an N-pulse exists only if, for each k, $\delta_k^{(1)}$ and $\delta_k^{(2)}$ take *opposite* signs. This means that the kth and the $(k+1)$th excursions follow *different* primary orbits from the set $\{S\overline{q}(t)\colon S\in G\}$. Next, we would like to apply the implicit function theorem (IFT) at $\mu=0$ and $T=\infty$, because $T=\infty$ corresponds to all the T_k being ∞, which simply represents the known 1-pulse solution $\overline{q(t)}$. In order to do so, we need to define a new variable r such that $r=0$ will correspond to $T=\infty$. Before introducing the appropriate transformation, we remark that the solvability of Eq. (11) hinges on the following fact. The transcendental equation

$$-1+c(\nu)r^{\nu}+O(r^{\omega})=0, \quad \omega>0, \quad \nu>0, \tag{12}$$

where $c(\nu)$ is a smooth function with $c(0)>1$, does not have (small) positive solutions $r(\nu)$ when $\nu\leqslant 0$; whereas for $\nu>0$, Eq. (12) can be solved by setting $\rho=r^{\nu}$ and applying the IFT at $\nu=0$ and $\rho_0=1/c(0)$. Note that, for the remainder term to vanish, we require $\rho^{\omega/\nu}\to 0$ as $\nu\downarrow 0$ (with $\omega>0$ fixed), and this is certainly true if $\rho(0)<1$, i.e., $c(0)>1$.

With this information in mind, if $|b_1(0)|<|b_2(0)|$, defining

$$c(\mu)=\frac{b_2(\mu)^2(1+\mu)^2}{b_1(\mu)^2}, \quad r=e^{-2T}, \quad a_j=e^{-2(T_j-T)},$$

leads to a simplified equation

$$(a_k r)[-1+c(\mu)(a_k r)^{\mu}]+O(r^{1+\omega})=0,$$

which is solvable, by taking $\rho_k(\mu)=(a_k r)^{\mu}$ and applying the IFT, if and only if $\mu>0$, i.e., $\theta_1<\theta_2$.

On the other hand, if $|b_1(0)|>|b_2(0)|$, letting

$$c(\mu)=\frac{b_1(\mu)^2}{b_2(\mu)^2(1+\mu)^2},$$

$$r=e^{-2(1+\mu)T}, \quad a_j=e^{-2(1+\mu)(T_j-T)},$$

gives a simplified equation

$$(a_k r)[-1+c(\mu)(a_k r)^{1/(1+\mu)-1}]+O(r^{1+\omega})=0,$$

which is solvable, by taking $\rho_k(\mu)=(a_k r)^{1/(1+\mu)-1}$ and applying the IFT, if and only if $1/(1+\mu)-1>0$, i.e., $\mu<0$ or equivalently $\theta_1>\theta_2$.

In summary, we make the following observations.

(i) For N-pulses to exist, $\delta_k^{(1)}$ and $\delta_k^{(2)}$ cannot be of the same sign. This means that one of $\delta_k^{(1)},\delta_k^{(2)}$ has to be -1, i.e., an N-pulse must be of *alternating* type, as following the same primary orbit during consecutive excursions would necessitate $\delta_k^{(1)}=\delta_k^{(2)}=+1$. At the same time, an N-pulse cannot alternate in all of its components, because one of $\delta_k^{(1)},\delta_k^{(2)}$ also needs to be $+1$.

(ii) The direction of bifurcation is determined by the relative sizes of $b_1(0)$ and $b_2(0)$, information which comes from the primary solution $\overline{q(t)}$. If $|b_1(0)|\neq|b_2(0)|$, the bifurcation is one sided, as shown above. If $|b_1(0)|=|b_2(0)|$, it is possible for N-pulses to exist at $\theta_1=\theta_2$ and on both sides of this line in parameter space; more delicate analysis is required in this case.

(iii) If multihumped solutions are generated from this homoclinic bifurcation, then there is no limit on the number of humps N that they may possess. In addition, it is possible to show uniqueness of an N-pulse, for any given N and small enough μ, up to symmetry of the system [14].

IV. SPECTRAL ANALYSIS

To determine the stability of a real-valued solitary-wave solution $u=\Phi(t)$ of Eq. (1), one first needs to study the spectrum of the linearization (6),

$$\mathcal{L}=\mathcal{J}\begin{pmatrix} L_R & 0 \\ 0 & L_I \end{pmatrix},$$

which depends on Φ. The essential spectrum of such an operator lies entirely on the imaginary axis; also, the discrete spectrum is symmetric with respect to both real and imaginary axes. Therefore, existence of any discrete spectrum off the imaginary axis would imply instability. Operators like Eq. (6) are usually not straightforward to analyze, even when $\Phi(t)$ is the primary pulse, or ground state. In this section we show that by dealing with the simpler self-adjoint operators L_R and L_I, and then putting this information together using a criterion developed by Jones and Grillakis, the instability of N-pulses can be demonstrated rigorously.

A sufficient condition, as stated by Jones [15], for the instability of pulses in nonlinear Schrödinger-type equations is as follows: if $n(L_R)-n(L_I)\neq 0,1$, then $\mathcal{L}$ has a real positive eigenvalue, and thus the pulse is unstable. By the theory developed by Alexander, Gardner, and Jones [16], the spectrum of a linearized operator such as L_R, L_I, or $\mathcal{L}$ evaluated at an N-pulse resembles that of the same operator evaluated at the 1-pulse, with each eigenvalue copied N times but slightly perturbed from its original spot, i.e., each single eigenvalue splits into N. Since in our case $n(L_R)$ is already greater than $n(L_I)$ for the primary pulse, the stability of bifurcating N-pulses will depend crucially on how the critical eigenvalues at zero split. Note that for each of L_R and L_I, one eigenvalue will always stay at zero, by virtue of the translation and phase invariances. Assuming that at the primary pulse L_R and L_I have the same number of eigenvalues at zero, then instability occurs if either of the following is true: (i) for L_R, at least one of the critical eigenvalues splits to the negative side, or (ii) for L_I, at least one of the critical eigenvalues splits to the positive side; both these possibilities will ensure that the imbalance between negative eigenvalues is greater than 1. These conditions can also be readily derived from Grillakis's results [17], which imply that the sum of the numbers of negative critical eigenvalues for L_R and positive critical eigenvalues for L_I gives the number of $\pm$ pairs of real eigenvalues for $\mathcal{L}$.

Sandstede [18] established that the near-zero critical eigenvalues of a linearized operator L, evaluated at an N-pulse, are determined by the zeros of

$$E(\lambda)=\det[A-\lambda MI+R(\lambda)],$$

where A is an $N\times N$ tridiagonal matrix, M is a Melnikov-type integral, and $R(\lambda)$ is a remainder term [18]. In the case of a reversible system, A is in addition symmetric, and moreover the signs of its eigenvalues have a direct relationship to the signs of its leading order entries, which are determined by a sequence $\{\tilde{a}_i : 1\leqslant i\leqslant N-1\}$. This information, combined with the sign of M, will give us the signs of the critical eigenvalues of L. The following quantities therefore need to be calculated:

$$M=\int_{-\infty}^{\infty}\langle\Psi(t,0),B\Upsilon(t,0)\rangle\,dt,$$

$$\tilde{a}_i=\langle\Psi_{i+1}(-T_i,\mu_N),\Upsilon_i^+(T_i,\mu_N)\rangle,$$

where $\Upsilon(t,\mu)$ solves the zero-eigenvalue problem $L\vec{Z}=0$, expressed as a first-order ODE, at parameter value μ, $\Psi(t,\mu)$ solves the associated adjoint ODE, and B is a matrix that arises from writing the general eigenvalue problem $L\vec{Z}=\lambda\vec{Z}$ in the form of a first-order system $Y'=[\mathrm{D}_u f(u(t),\mu)+\lambda B]Y$; "+" refers to a solution in the stable manifold of the equilibrium point, and μ_N denotes the parameter value at which the N-pulse exists. It is not hard to show that if L is a second-order self-adjoint linear differential operator, then $\Psi=J^{-1}\Upsilon$ (where J was defined in Sec. II), and

$$B=\begin{pmatrix}0 & 0\\ -I & 0\end{pmatrix}.$$

Thus

$$M=\int_{-\infty}^{\infty}\langle\Psi(t,0),BJ\Psi(t,0)\rangle\,dt$$
$$=\int_{-\infty}^{\infty}\left\langle\Psi(t,0),\begin{pmatrix}0 & 0\\ 0 & -I\end{pmatrix}\Psi(t,0)\right\rangle dt<0.$$

For L_R, we have

$$\Upsilon=q'=J\nabla H(q)=J\nabla^2H(0)q+O(|q|^2)$$

so $\Psi=\nabla^2H(0)q+O(|q|^2)$. Hence, as in Sec. III, using reversibility, symmetry, and persistence of the 1-pulse for $\mu\neq 0$, we derive [19]

$$\tilde{a}_i=\langle\kappa_i\{\nabla^2H(0)(-J)\nabla^2H(0)R\}\kappa_{i+1}q(T_i),q(T_i)\rangle+O(|q|^3),$$

where all quantities are evaluated at $\mu=\mu_N$, and $\kappa_i,\kappa_{i+1}\in G$. Using the descriptions of $\nabla^2H(0)$ and R contained in Sec. II we can further calculate

$$\tilde{a}_i=\left\langle\kappa_i\begin{pmatrix}0 & -\Theta\\ -\Theta & 0\end{pmatrix}\kappa_{i+1}q(T_i),q(T_i)\right\rangle+O(|q|^3).$$

We saw that the kernel of L_I contains $\Gamma\Phi$, so for L_I we have

$$\Upsilon=\begin{pmatrix}\Gamma\Phi\\ \Gamma\Phi'\end{pmatrix}=\begin{pmatrix}\Gamma & 0\\ 0 & \Gamma\end{pmatrix}q$$

since $q=(\Phi,\Phi')$. This allows us to obtain

$$\tilde{a}_i=\left\langle\kappa_i\begin{pmatrix}0 & \Gamma^2\\ \Gamma^2 & 0\end{pmatrix}\kappa_{i+1}q(T_i),q(T_i)\right\rangle.$$

Note that the leading order terms of the $\tilde{a}_i$ for L_R and L_I are of very similar form, but there is an important sign difference in the matrices that appear in their expressions. This, together with the fact that the M's for L_R and L_I will both be negative, implies that, at an N-pulse, the critical eigenvalues for L_R have an *opposite* distribution from those of L_I; thus we need to determine the distribution of eigenvalues for only one of these operators.

For instance, let us choose L_I to study. From Eq. (5), we see that $q(T_i)\sim e^{-\lambda^s(\mu_N)T_i}$, where $\lambda^s(\mu_N)$ is the leading eigenvalue at $\mu=\mu_N$. Assuming Eq. (4), we then obtain

$$\tilde{a}_i=2\Gamma^2d_id_{i+1}\Phi(T_i)\cdot\Phi'(T_i),$$

where Γ^2 is a diagonal matrix with positive entries and $\Phi(T_i,\mu_N)\cdot\Phi'(T_i,\mu_N)<0$ to leading order. Since $M<0$, the N-pulse will be unstable if the leading order terms of $\tilde{a}_i$ contribute a negative sign for at least one i. Thus, for at least one i, d_id_{i+1} has to contribute a *positive* sign to the leading order component in order to satisfy the criterion for instability. If (as assumed in Sec. II) the d_i are diagonal with ± 1 as their nonzero entries, then d_id_{i+1} contributes a positive sign at leading order if and only if the leading order component does not alternate during the ith and $(i+1)$th excursions. From Sec. III we saw that N-pulses must alternate, but not in every component; here we further see that an N-pulse is stable only when it is the leading order component that exhibits strict alternation. Compare this with results for the scalar phase-sensitive amplifier equation [13], which has only one component: although it does exhibit nonalternating multipulses, only the strictly alternating N-pulses are stable.

V. EXAMPLES

The normalized equations [5,6] describing the interaction between a fundamental frequency wave and its second harmonic in dispersive quadratic ($\chi^{(2)}$) media can be written as

$$i\frac{\partial w}{\partial z}+r\frac{\partial^2 w}{\partial t^2}-\theta w+w^*v=0, \tag{13}$$

$$i\sigma\frac{\partial v}{\partial z}+s\frac{\partial^2 v}{\partial t^2}-\alpha v+\frac{1}{2}w^2=0.$$

This model is valid for the temporal (pulses in optical fibers) case within a suitable coordinate frame, and in the spatial (beams in slab waveguides) case when "walk off" is neglected. Here $w(z,t)$ and $v(z,t)$ represent the envelope amplitudes of the fundamental and second-harmonic waves, respectively; we have $r,s=\pm 1$, $\sigma>0$ in the temporal case and $\sigma=2$ in the spatial case, and θ, α are real parameters which we will take to be positive; α incorporates the wave-vector mismatch between the two harmonics, and without loss of generality θ can be scaled to 1. We shall consider only the "bright-bright" $r=s=+1$ case in this article. Clearly $u_1=w$, $u_2=v$, $\theta_1=1$, $\theta_2=\alpha$, and we let $\mu=\sqrt{\alpha}-1$.

The ODEs describing the behavior of real-valued steady-states

$$w'=p_w,\quad v'=p_v, \tag{14}$$

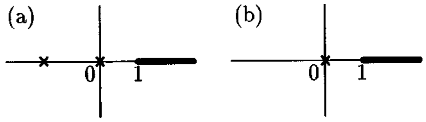

FIG. 2. Spectra of (a) L_R and (b) L_I at the 1-pulse. The crosses represent simple eigenvalues, and the thick line segments depict essential spectra.

$$p_w' = w - wv, \quad p_v' = \alpha v - \tfrac{1}{2} w^2$$

are invariant under the reflection $S:(w,v,p_w,p_v) \mapsto (-w,v,-p_w,p_v)$, so we take $G=\{I,S\}$. At $\mu=0$, the eigenvalues of the linearization of Eq. (14) are in resonant semisimple configuration, and there is an explicitly known 1-pulse $\overline{q} = (\overline{w},\overline{v},\overline{w}',\overline{v}')$ with $\overline{w}(t) = (3/\sqrt{2})\mathrm{sech}^2(t/2)$ and $\overline{v}(t) = \frac{3}{2}\mathrm{sech}^2(t/2)$. Nondegeneracy of this primary solution can be shown via a diagonalization procedure. The N-pulses we seek are solutions that closely follow $\overline{q}(t)$ and $S\overline{q}(t)$ in some order $\{\kappa_j\}_{j=1,\ldots,N}$, where $\kappa_j \in \{I,S\}$. Observe that $\kappa_j\kappa_{j+1} = \mathrm{diag}(\delta_j^{(1)},\delta_j^{(2)},\delta_j^{(1)},\delta_j^{(2)})$ where here $\delta_j^{(1)} = \pm 1$ and $\delta_j^{(2)} = +1$ for all j. Using Eq. (5) and the explicit expansion of $\overline{q}(t)$ at $\mu=0$, we deduce $b_1(0) = 6\sqrt{2}$ and $b_2(0)=6$. Since $|b_1(0)| > |b_2(0)|$, the analysis in Sec. III tells us that N-pulses exist for $\mu<0$, i.e., $\alpha<1$ $(=\theta)$, but not for $\mu \geqslant 0$ or $\alpha \geqslant \theta$. Moreover, these N-pulses must have $\delta_j^{(1)} = -1$ for all j. In other words, the multipulses are constructed by concatenating, alternately, a 1-pulse with positive w component and a 1-pulse with negative w component; they exhibit strict alternation in their w components, but do not alternate in their v components.

Turning to stability, we first check that the properties described in Sec. II are satisfied. Equations (13) are invariant under the phase transformation $(w,v) \mapsto (we^{is}, ve^{2is})$, so $\Gamma = \mathrm{diag}(1,2)$. The spectra of L_R and L_I at the primary pulse are shown in Fig. 2. They were determined after diagonalizing L_R and L_I, and we point out that the diagonalization of these operators is possible because Eq. (13) has a pure-power nonlinearity, and also the components of the 1-pulse are constant multiples of each other (in this example $\overline{w} = \sqrt{2}\overline{v}$). Now since the N-pulses are found for $\alpha<1$, the leading eigenvalues are $\pm\sqrt{\alpha} = \pm\sqrt{\theta_2}$ and, referring to the eigenvectors $\epsilon_j^{\pm}$ given in Sec. II, we see that the v-component is the leading order one. Because there is no alternation in v, the analysis of Sec. IV implies that all the multipulses are unstable. In fact, we find that for N-pulses, all but one of the critical eigenvalues of L_R split to the negative side and, to mirror this, all but one of the critical eigenvalues of L_I split to the positive side. We therefore have a "worst possible case" of instability, which appears to stem from both the translation and phase symmetries (manifested through L_R and L_I, respectively).

A system of the following form has been used [1,2] to model pulse propagation in birefringent $\chi^{(3)}$ (Kerr) media:

$$i\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial t^2} - \alpha u + (a|u|^2 + b|v|^2)u = 0, \tag{15}$$

$$i\frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial t^2} - \beta v + (a|v|^2 + b|u|^2)v = 0.$$

When $b=a$, the system is known as the Manakov equations, and is integrable; the stability of its 1-pulse solitons under small Hamiltonian perturbations has been investigated in [20]. Real-valued, z-stationary, solutions of Eq. (15) solve a system of Hamiltonian ODEs, which are invariant under the group of symmetries $G = \{I, \mathrm{diag}(-1,1,-1,1), \mathrm{diag}(1,-1,1,-1), -I\}$. An explicit 1-pulse exists at $\alpha=\beta$, with $\overline{u}(t) = \overline{v}(t) = \sqrt{2\alpha/(a+b)}\mathrm{sech}(\sqrt{\alpha}t)$. In addition, the equations also admit pulses that have a trivial u or v component, but we shall not be concerned with such solutions here. Numerical evidence for the existence of multipulses has been presented in [1]. We are interested in showing analytically the existence and stability of N-pulses that may bifurcate from the aforementioned explicit 1-pulse upon varying α (or β) near $\alpha=\beta$. Now both $\delta_j^{(1)}$ and $\delta_j^{(2)}$ may take either $+1$ or -1 as values; if bifurcation of N-pulses is to occur, $\delta_j^{(1)},\delta_j^{(2)}$ should be of opposite sign. We also have $b_1(0)=b_2(0)$ in this case (because $\overline{u}=\overline{v}$), which is a rather degenerate situation not treated in Sec. III. Nevertheless, the same techniques can be used to analyze this situation, at least partially; they enable us to show that there are multipulses with two or three humps, and that these multihumped pulses are unstable. Details of these proofs, as well as the more complicated analysis for multipulses with more than three humps, will be presented elsewhere [21].

VI. CONCLUSIONS

We introduced analytical methods for constructing multihumped solitary-waves, and for studying their stability, in systems of coupled NLS equations. We identified a specific mechanism whereby multipulses can be generated, namely, a homoclinic bifurcation occurring at a resonant semisimple eigenvalue scenario. A stability analysis was then undertaken, and the features of certain multipulses that lead to their instability were pointed out. It appears that the very conditions that guarantee the existence of multihumped pulses also imply that the waves will be unstable. We have demonstrated that instabilities related to translation and phase symmetries of the system will always occur together; furthermore, from the instability criterion given in Sec. IV, it would seem that the likelihood of either kind of instability occurring is rather high.

These techniques and observations were applied to two examples from nonlinear optics that have received much attention in recent years. The existence and instability of multipulses was proved for both systems.

Since the eigenvalues associated with translational and phase invariance are all unstable, this opens up the possibility of using the multihumped pulses for all-optical steering or switching devices. Indeed, the instabilities are related to changes in the amplitudes of the individual humps in the pulse train as well as to changes in their relative position. The issue is then to determine which of these instabilities is dominant. We believe that the analytical methods introduced here can also be used to decide this question; we will report on this elsewhere.

We emphasize that the multihumped pulses might exhibit unstable eigenvalues other than those described here. Indeed, there is the possibility of additional unstable eigenvalues being created near radiation modes [22,23].

It appears that the phase invariance of the underlying equation is responsible for the instability of the multihumped pulses. If the phase invariance is broken, for instance by using phase-sensitive amplification, it is possible that multihumped pulses are stable [13]. Stable pulses have also been observed when the distances between consecutive pulses in the pulse train become relatively small [24].

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